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Program Notes

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Program Notes

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Contents

1 Introduction

1.1 Continuum Dynamic Damage Model

1. Dynamical equilibrium equations
$$\boldsymbol{\nabla} \cdot \dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{F}} = \rho \ddot{\boldsymbol{u}} \tag{1}$$

2. Gepmetrical equations (Strain-displacement relations)
$$\dot{\boldsymbol{\varepsilon}} = \boldsymbol{\nabla} \cdot \dot{\boldsymbol{u}} \tag{2}$$

3. Constitutive equations (Stress-strain relations)
$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{el} + \dot{\boldsymbol{\varepsilon}}^{ed} + \dot{\boldsymbol{\varepsilon}}^{id} = \dot{\boldsymbol{\varepsilon}}^{E} + \dot{\boldsymbol{\varepsilon}}^{id} \tag{3}$$
$$\dot{\boldsymbol{\sigma}} = \mathbb{D}_{ed} : \dot{\boldsymbol{\varepsilon}} \tag{4}$$

1.2 Constitutive equations of the skeleton

1.2.1 State equations of the skeleton

The constitutive damage models of isothermal media account for $\boldsymbol{\varepsilon}^E$ and $\boldsymbol{\Omega}$ as the state variables related to the matrix. Based on the postulate of local state, the free energy of the skeleton can be generally expressed in the form:
$$\psi_s = \psi_s(\boldsymbol{\varepsilon}^E, \boldsymbol{\Omega}) \tag{5}$$
The variables $\boldsymbol{\varepsilon}^E$ and $\boldsymbol{\Omega}$ are subset of external state variables.
The Inequality of Clausisus-Duhem is derived from the combination of the two first laws of thermodynamics. the energy dissipated in each step is:
$$\dot{\Phi}_s = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi}_s \geq 0 \tag{6}$$
The free energy rate writes:
$$\dot{\psi}_s = \frac{\partial \psi_s}{\partial \boldsymbol{\varepsilon}^E} : \dot{\boldsymbol{\varepsilon}}^E + \frac{\partial \psi_s}{\partial \boldsymbol{\Omega}} : \dot{\boldsymbol{\Omega}} \tag{7}$$
The dissipation potential of solid skeleton should satisfy the Clausis-Duhem Inequality:
$$\dot{\Phi}_s = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \psi_s}{\partial \boldsymbol{\varepsilon}^E} : \dot{\boldsymbol{\varepsilon}}^E - \frac{\partial \psi_s}{\partial \boldsymbol{\Omega}} : \dot{\boldsymbol{\Omega}} \tag{8}$$
$$= \left(\boldsymbol{\sigma} - \frac{\partial \psi_s}{\partial \boldsymbol{\varepsilon}^E} \right) \boldsymbol{\varepsilon}^E + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{id} - \frac{\partial \phi_s}{\partial \boldsymbol{\Omega}} : \dot{\boldsymbol{\Omega}} \geq 0 \tag{9}$$

Since the variables can vary irrespective of the others and the inequality should always hold. If the internal variables do not vary, it can be concluded that:
$$\boldsymbol{\sigma} = \frac{\partial \psi_s}{\partial \boldsymbol{\varepsilon}^E}; \quad \boldsymbol{Y} = -\frac{\partial \psi_s}{\partial \boldsymbol{\Omega}} \tag{10}$$
Equation 10 asociate the state variables $\boldsymbol{\varepsilon}^E$ and $\boldsymbol{\Omega}$ to their conjugate thermodynamic state variables $\boldsymbol{\sigma}$ and $\boldsymbol{Y}$. So the energy dissipation can be rewritten as:
$$\dot{\Phi}_s = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{id} + \boldsymbol{Y} : \dot{\boldsymbol{\Omega}} \tag{11}$$
$$= \dot{\Phi}_{id} + \dot{\Phi}_d \geq 0 \tag{12}$$

1.3 Flow rule

1.3.1 Kuhn-Tucker condition

In non-associated flow law for damage evolution, a new dissipation potential function $g(\boldsymbol{Y})$ is introduced to consider the growth of damage. The other side, the damage function, $f(\boldsymbol{Y}, \boldsymbol{\Omega})$, is used as the potential to derive the flow rule for irreversible strain. $\boldsymbol{Y}$ is a function of stress, $\boldsymbol{\sigma}$, so the potential is also the function of stress, $f(\boldsymbol{\sigma}, \boldsymbol{\Omega})$. Here, for the simplification, assume the damage potential $g(\boldsymbol{Y})$ is independent of stress and the damage function is independent of damage driving force. The material tries to dissipate energy in the easiest way and maximize dissipation. The problem converts to maximize the $\dot{\Phi}_s$ under the constrain $g = 0$ and $f = 0$, which is also called Kuhn-Tucker condition. In order to optimize the function, Lagrange Multiplier $\dot{\lambda}_d$ and $\dot{\lambda}_{id}$ is taken into account to form a new function:

$$\mathscr{F}(\boldsymbol{\sigma}, \boldsymbol{Y}, \dot{\lambda}_d, \dot{\lambda}_{id}) = \dot{\Phi}_s - \dot{\lambda}_d\, g(\boldsymbol{Y}) - \dot{\lambda}_{id}\, f(\boldsymbol{\sigma}, \boldsymbol{\Omega}) \tag{13}$$

The necessary conditions for this optimization problem are:

$$\frac{\partial \mathscr{F}}{\partial \boldsymbol{\sigma}} = \frac{\partial \dot{\Phi}_s}{\partial \boldsymbol{\sigma}} - \dot{\lambda}_{id} \frac{\partial f}{\partial \boldsymbol{\sigma}} = 0, \qquad \frac{\partial \mathscr{F}}{\partial \boldsymbol{Y}} = \frac{\partial \dot{\Phi}_s}{\partial \boldsymbol{Y}} - \dot{\lambda}_d \frac{\partial g}{\partial \boldsymbol{Y}} = 0 \tag{14}$$

which results into:

$$\dot{\boldsymbol{\varepsilon}}^{id} = \dot{\lambda}_{id} \frac{\partial f}{\partial \boldsymbol{\sigma}}, \qquad \dot{\boldsymbol{\Omega}} = \dot{\lambda}_d \frac{\partial g}{\partial \boldsymbol{Y}} \tag{15}$$

We need two equations to solve for the lagrangian multiplier $\dot{\lambda}_d$ and $\dot{\lambda}_{id}$. The Kuhn-Tucker condition for damage criterion gives one:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{Y}} : \dot{\boldsymbol{Y}} + \frac{\partial f}{\partial \boldsymbol{\Omega}} : \dot{\boldsymbol{\Omega}} = \frac{\partial f}{\partial \boldsymbol{Y}} : \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \boldsymbol{\Omega}} : \dot{\boldsymbol{\Omega}} = 0 \tag{16}$$

Assume the damaged stiffness is constant in each increment of iteration. The stress increment can be derived by the following equation which can relates to the irreversible strain.

$$\dot{\boldsymbol{\sigma}} = \mathbb{D}_e : \dot{\boldsymbol{\varepsilon}}^E = \mathbb{D}_e : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^{id}) \tag{17}$$

For single Lagrangian multiplier,

$$\dot{\lambda}_d = \frac{\frac{\partial f}{\partial \boldsymbol{Y}} : \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{\sigma}} : \mathbb{D}_e : \dot{\boldsymbol{\varepsilon}}}{\frac{\partial f}{\partial \boldsymbol{Y}} : \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{\sigma}} : \mathbb{D}_e : \frac{\partial f}{\partial \boldsymbol{\sigma}} - \frac{\partial f}{\partial \boldsymbol{\Omega}} : \frac{\partial g}{\partial \boldsymbol{Y}}} \tag{18}$$

From Kuhn-Tucker condition:

$$a_{11}\, \dot{\lambda}_d + a_{12}\, \dot{\lambda}_{id} = b_1 \tag{19}$$

where

$$a_{11} = -\frac{\partial f}{\partial \boldsymbol{\Omega}} : \frac{\partial g}{\partial \boldsymbol{Y}} \tag{20}$$

$$a_{12} = \frac{\partial f}{\partial \boldsymbol{Y}} : \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{\sigma}} : \mathbb{D}_e : \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{21}$$

$$b_1 = \frac{\partial f}{\partial \boldsymbol{Y}} : \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{\sigma}} : \mathbb{D}_e : \dot{\boldsymbol{\varepsilon}} \tag{22}$$

To do the iteration, $\dot{\lambda}_{id}$ is given as the initial condition, so $\dot{\lambda}_d$ can be obtained:

$$\dot{\lambda}_d = \frac{\frac{\partial f}{\partial \boldsymbol{Y}} : \frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{\sigma}} : \mathbb{D}_e : \left(\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}_{id} \frac{\partial f}{\partial \boldsymbol{\sigma}}\right)}{-\frac{\partial f}{\partial \boldsymbol{\Omega}} : \frac{\partial g}{\partial \boldsymbol{Y}}} \tag{23}$$

The second equation can be obtained by the law of energy conservation

1.3.2 Energy dissipation

The work by the external force is $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$ in each step, so the accumulate work is $\int \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \,\mathrm{d}t$. The accumulate strain energy stored in the solid body is $\frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}^E$, and the dissipated energy is the differential between the the differential between external work and recoverable strain energy. The mechanical energy lost, $\int \dot{\Phi}_d \,\mathrm{d}t$, in the loading is the area displayed in figure 1 with pink color and should be always non-negative, while the recoverable elastic strain energy is the area of the triangle as figure 1 with light green color.

$$\int \dot{\Phi}_d \,\mathrm{d}t = \int \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \,\mathrm{d}t - \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}^E \geq 0 \tag{24}$$

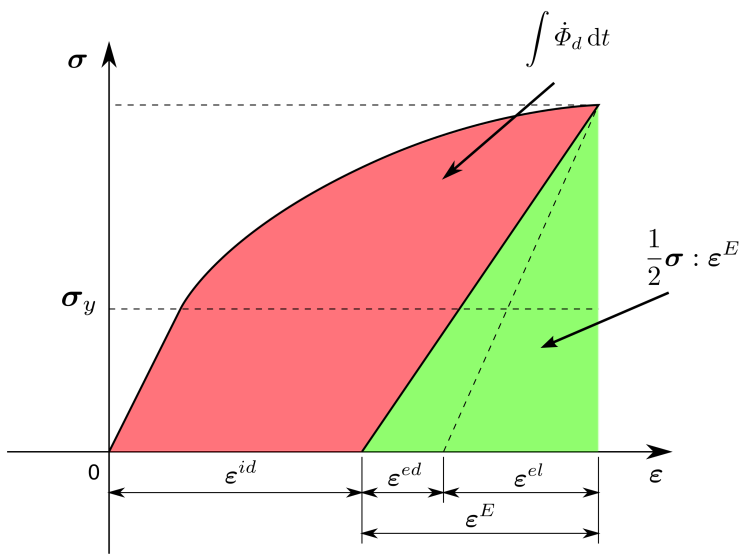

Figure 1: Energy spent in one loading process

The energy dissipated by damage is concluded into two means, crack generating without additional deformation and debonding paris of crack surfaces. Both of these two approaches can be represented by the mechanical response. Crack generation decrease the stiffness of material, which can increase the capacity that material stores the energy; while debonding is related to non-linear strains. Since there are no other ways of energy transfer (thermal transport or radiation), the energy should be conserved in total. Based on response of material, all of the work by external loading should be splitted into three parts as figure 2 shows (The white part of energy is compensated by the extra part of the energy, related to irreversible strain, above the stress curve). It is also true from the strain decomposition in another side. The energy conservation is expressed as:

$$\int \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \,\mathrm{d}t = \int \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{el} \,\mathrm{d}t + \int \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{ed} \,\mathrm{d}t + \int \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{id} \,\mathrm{d}t \tag{25}$$

There are two parts of the energy related to elasto-damage strain, $\boldsymbol{\varepsilon}^{ed}$. Not both of the elasto-damage strain is dissipated, some of them is stored in the solid body as strain energy ($\frac{1}{2}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{ed}$); the other is dissipated (if material is unloaded), which won't induce the any strain increment. This part noted as $\int \boldsymbol{Y} : \dot{\boldsymbol{\Omega}} \,\mathrm{d}t$ is due to the crack generating but no displacement (figure 3).

$$\int \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{ed} \,\mathrm{d}t = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}^{ed} + \int \boldsymbol{Y} : \dot{\boldsymbol{\Omega}} \,\mathrm{d}t \tag{26}$$

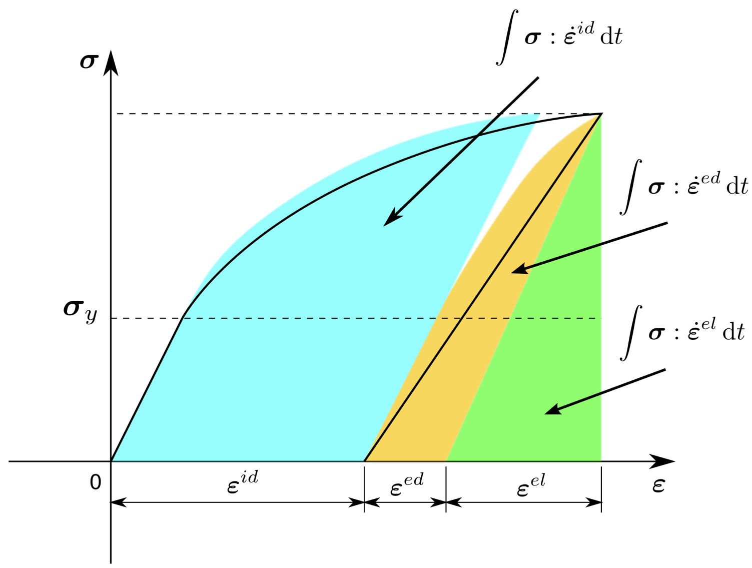

Figure 2: Energy splitting in one loading process

The energy related to debonding of the crack surfaces will be dissipated ($\int \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{id}\,\mathrm{d}t$ related to irreversible strain) or stored ($\frac{1}{2}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{ed}$, one part of energy related to elasto-damage strain.)

Based on the mechanical part, the lost of the energy is dissipated by the irreversible strain and crack generation due to damage.

$$\int \dot{\Phi}_d\,\mathrm{d}t = \int \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{id}\,\mathrm{d}t + \int \boldsymbol{Y} : \dot{\boldsymbol{\Omega}}\,\mathrm{d}t \geq 0 \tag{27}$$

Therefore, the following equation is hold:

$$\int \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}\,\mathrm{d}t - \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}^{E} = \int \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{id}\,\mathrm{d}t + \int \boldsymbol{Y} : \dot{\boldsymbol{\Omega}}\,\mathrm{d}t \tag{28}$$

If in the FEM, it could be written in the summation of all increments.

$$\sum \boldsymbol{\sigma}_{\theta n} : \dot{\boldsymbol{\varepsilon}}_n - \frac{1}{2}\boldsymbol{\sigma}_n : \boldsymbol{\varepsilon}_n^{E} = \sum \boldsymbol{\sigma}_{\theta n} : \dot{\boldsymbol{\varepsilon}}_n^{id} + \sum \boldsymbol{Y}_{\theta n} : \dot{\boldsymbol{\Omega}}_n \tag{29}$$

Where $\boldsymbol{\sigma}_{\theta n}$ and $\boldsymbol{Y}_{\theta n}$ are the resultant stress and damage force during each increment when calculating the energy change:

$$\boldsymbol{\sigma}_{\theta n} = (1-\theta)\boldsymbol{\sigma}_{n-1} + \theta\,\boldsymbol{\sigma}_n \tag{30}$$

$$\boldsymbol{Y}_{\theta n} = (1-\theta)\boldsymbol{Y}_{n-1} + \theta\,\boldsymbol{Y}_n \tag{31}$$

θ is the approximation factor. The total elastic strain can be decomposed into:

$$\boldsymbol{\varepsilon}_n^{E} = \boldsymbol{\varepsilon}_{n-1}^{E} + \dot{\boldsymbol{\varepsilon}}_n^{E} = \boldsymbol{\varepsilon}_{n-1}^{E} + \dot{\boldsymbol{\varepsilon}}_n - \dot{\boldsymbol{\varepsilon}}_n^{id} \tag{32}$$

The equation 29 is reorganized as:

$$\begin{aligned} &\left(\boldsymbol{\sigma}_{\theta n} - \frac{1}{2}\boldsymbol{\sigma}_n\right) : \dot{\boldsymbol{\varepsilon}}_n^{id} + \boldsymbol{Y}_{\theta n} : \dot{\boldsymbol{\Omega}}_n \\ =&\sum \boldsymbol{\sigma}_{\theta n} : \dot{\boldsymbol{\varepsilon}}_n - \frac{1}{2}\boldsymbol{\sigma}_n : (\boldsymbol{\varepsilon}_{n-1}^{E} + \dot{\boldsymbol{\varepsilon}}_n) - \sum \boldsymbol{\sigma}_{\theta n-1} : \dot{\boldsymbol{\varepsilon}}_{n-1}^{id} - \sum \boldsymbol{Y}_{\theta n-1} : \dot{\boldsymbol{\Omega}}_{n-1} \end{aligned} \tag{33}$$

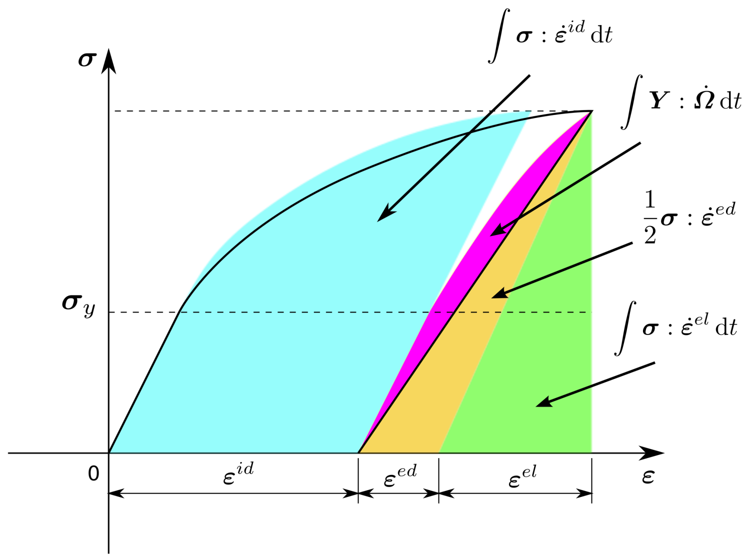

Figure 3: Energy splitting in one loading process

From the conservation of energy, the second equation for Lagrangian Multipliers can be obtained.

$$a_{21}\,\dot{\lambda}_d + a_{22}\,\dot{\lambda}_{id} = b_2 \tag{34}$$

where

$$a_{21} = \boldsymbol{Y}_{\theta n} : \frac{\partial g}{\partial \boldsymbol{Y}} \tag{35}$$

$$a_{22} = \left(\boldsymbol{\sigma}_{\theta n} - \frac{1}{2}\boldsymbol{\sigma}_n\right) : \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{36}$$

$$b_2 = \sum \boldsymbol{\sigma}_{\theta n} : \dot{\boldsymbol{\varepsilon}}_n - \frac{1}{2}\boldsymbol{\sigma}_n : (\boldsymbol{\varepsilon}^E_{n-1} + \dot{\boldsymbol{\varepsilon}}_n) - \sum \boldsymbol{\sigma}_{\theta n-1} : \dot{\boldsymbol{\varepsilon}}^{id}_{n-1} - \sum \boldsymbol{Y}_{\theta n-1} : \dot{\boldsymbol{\Omega}}_{n-1} \tag{37}$$

$$\begin{Bmatrix} \dot{\lambda}_d \\ \dot{\lambda}_{id} \end{Bmatrix} = \frac{1}{\Delta}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \tag{38}$$

where

$$\Delta = a_{11}a_{22} - a_{12}a_{21} \tag{39}$$